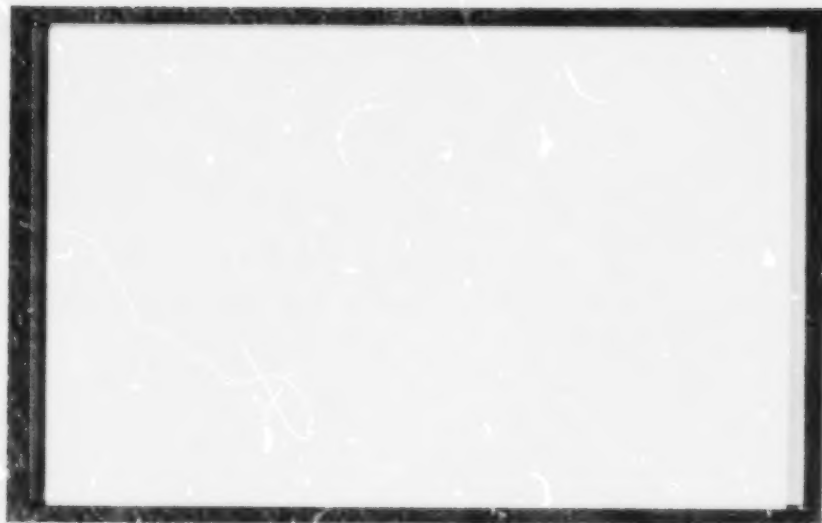


UNIVERSITÉ DE MONTRÉAL



CENTRE DE RECHERCHE SUR LES TRANSPORTS



CASE POSTALE 6128, SUCCURSALE A, MONTRÉAL, QUÉBEC, H3C 3J7

SOME FURTHER RESULTS ON k-STAR
CONSTRAINTS AND COMB INEQUALITIES

by

Gilbert Laporte*
and Jean-Marie Bourjolly**

* École des Hautes Études Commerciales,
5255, avenue Decelles, Montréal, Québec, CANADA H3T 1V6

** Department of Combinatorics and Optimization,
University of Waterloo, Waterloo, Ontario, CANADA N2L 3G1

Université de Montréal

Centre de recherche sur les transports - Publication #387
École des Hautes Études Commerciales, Cahier du GERAD G-84-17

Novembre 1984

Dépôt légal - 1er trimestre 1985

Abstract

Some earlier results on k -star constraints and comb inequalities are generalized.

Résumé

On généralise certains résultats antérieurs sur les contraintes de type k -étoile et sur les inégalités de peigne.

1. INTRODUCTION

Consider a graph $G = (N, E, C)$ where $N = \{1, \dots, n\}$ is a set of nodes or cities, E is a set of undirected edges and $C = (c_{ij})$, a symmetrical matrix of non-negative distances associated with the edges (i,j) . Also consider the following four problems associated with G :

(TSP) The travelling salesman problem (see for example Dantzig et al. [1]).

Determine the shortest cycle passing through each node exactly once.

(RTSP) The road travelling salesman problem (Fleischmann [3]).

Determine the shortest cycle passing through each node at least once.

(CVRP) The capacitated vehicle routing problem (see for example Laporte and Nobert [9]).

To each node i is associated a non-negative weight or requirement d_i . It assumed that $d_1 = 0$. There are m vehicles, each having the same capacity D and based at city 1. These vehicles must be dispatched to the remaining $n-1$ cities in such a way that

(i) each city (except city 1) is visited exactly once by exactly one vehicle;

(ii) vehicle capacities are never exceeded by the sum of the weights of cities they visit;

(iii) the total distance travelled by all vehicles is minimized.

Here, m can be fixed a priori or considered as a decision variable.

(RCVRP) The road capacitated vehicle routing problem (Fleischmann [3]).

This problem is similar to the CVRP except that any city may now be visited more than once.

Several practical applications of these problems have been described in the Operational research literature. Their solution by exact methods for large values of n is therefore of prime interest. To this day, the most effective exact algorithms are based on integer linear programming (ILP) formulations involving $O(n^2)$ variables and making full use of constraint relaxation. In this context, considerable effort has been devoted in recent years to the generation of valid constraints (comb inequalities [6,7,10], k -star constraints [3], star constraints [4]) which are at first relaxed and generated only when they are found to be violated. By so doing, one hopes to make as little use as possible of branch and bound in the solution of the integer program. In fact, Fleischmann uses no branch and bound at all in his treatment of the RTSP [3].

The aim of this paper is to describe some recent results on k -star constraints and comb inequalities.

2. FORMULATIONS

We first present ILP formulations for the TSP, the RTSP and for the CVRP. These formulations do not seem to extend easily to the case of the RCVRP (see [3] and [8]); the fourth formulation provided only solves a relaxation (RCVRP') of the latter problem.

We define

x_{ij} : the number of times edge (i,j) is used in the optimal solution

$\bar{S} = N - S$ (for $S \subseteq N$)

$\lceil r \rceil$: the smallest integer greater than or equal to r if $r \geq 0$;
1 if $r < 0$

$\lfloor r \rfloor$: the largest integer less than or equal to r

$V(S)$: a lower bound on the number of vehicles required to visit city 1 and all cities of S (in the CVRP and in the RCVRP)

The objective function common to all these problems is

$$\text{minimize } \sum_{i < j} c_{ij} x_{ij}$$

The constraints are summarized in table 1.

TABLE 1
Summary of the constraints

Problem and reference	TSP [1]	RTSP [3]	CVRP [9]	RCVRP [3]
<u>Degree constraints</u>				
$\sum_{j=2}^n x_{1j}$	$= 2$	$= 2$	$= 2m$	$\geq 2m$
$\sum_{i < h} x_{ih} + \sum_{j > h} x_{hj}$ ($h \in N - \{1\}$)	$= 2$	$= 2$ and even	$= 2$	≥ 2 and even
<u>Connectivity constraints</u>				
$\sum_{i \in S, j \notin S} x_{ij}$ or $\sum_{i \notin S, j \in S} x_{ij}$	≥ 2 ($3 \leq S \leq n-3$; $S \subset N$)	$= 2$ ($2 \leq S \leq n-2$; $S \subset N$)	$\geq 2 V(S)$ ($3 \leq S \leq n-3$; $S \subset N - \{1\}$)	$\geq 2 V(S)$ ($2 \leq S \leq n-2$; $S \subset N - \{1\}$)
<u>Integrality constraints</u>				
m	not defined	not defined	≥ 0	≥ 0
x_{ij}	$= 0$ or 1 ($i < j$)	$= 0, 1$ or 2 ($i < j$)	$= 0, 1, 2$ ($i=1$) $= 0, 1$ ($1 < i < j$) not defined if $d_i + d_j > D$	≥ 0 and integer ($i < j$)

3. CONNECTIVITY CONSTRAINTS

Connectivity constraints prohibit the presence of disconnected components in the optimal solution, i.e. there will always exist a path between any pair of nodes. In the CVRP and the RCVRP, they further ensure that the solution contains no route whose total weight exceeds the vehicle capacity. These constraints have been discussed at length in previous papers (see for example [9]).

It is customary to define $V(S)$ as

$$V(S) = V_1(S) = \left\lceil \frac{\sum_{i \in S} d_i}{D} \right\rceil$$

but one can obtain a tighter bound by first solving a bin-packing problem on S and then defining $V(S)$ as

$$V(S) = V_2(S) = \text{the number of vehicles in the optimal bin-packing solution.}$$

The determination of $V_2(S)$ may require substantial effort as the bin-packing problem is NP-complete. However, earlier results obtained by Gavish [5] on the 13-city Dantzig and Ramser [2] example indicate that connectivity constraints are much tighter if $V_2(S)$ is used rather than $V_1(S)$. If the problems are to be solved by an algorithm which first relaxes connectivity constraints and generates them only when they are violated, the lower bounds on the objective function are likely to be much larger with $V_2(S)$ than with $V_1(S)$.

We now state some elementary properties of $V_1(S)$ and $V_2(S)$.

Property 1. $V_1(S) = V_2(S)$ if $|S| = 1$ or 2 .

Property 2. $V_1(S) \leq V_2(S) \leq \left\lceil \frac{2 \sum_{i \in S} d_i}{D} \right\rceil - 1$

The proof of this result is similar to that of the lemma in [9, p.79].

Property 3. For any S_1 and S_2 , $|V_1(S_1) + V_1(S_2)| - V_1(S_1 \cup S_2) = 0$ or 1.

However, the difference $|V_2(S_1) + V_2(S_2)| - V_2(S_1 \cup S_2)$ can take any positive integer value K . For example, let $|S_1| = |S_2| = 3K$, $D=3$, $d_i=1$ if $i \in S_1$, $d_i=2$ if $i \in S_2$. Then $V_2(S_1) = K$, $V_2(S_2) = 3K$ and $V_2(S_1 \cup S_2) = 3K$.

4. USING THE BIN-PACKING BOUND IN THE GENERATION OF k-STAR CONSTRAINTS

Fleischmann proved that the k-star inequalities developed for the RTSP (see theorem 2 in [3]) generalize to the RCVRP. Their right-hand side $3k+1$ is simply replaced by

$$2 \left\lceil \frac{k}{2} \sum_{\ell=1}^k [V_1(B_\ell) + V_1(C_\ell) + V_1(B_\ell \cup C_\ell)] \right\rceil$$

where the meaning of B_ℓ and C_ℓ is given in [3]. It is apparent that a tighter form of constraint could be obtained if V_1 was substituted by V_2 .

5. DERIVING NEW COMB INEQUALITIES FOR THE VEHICLE ROUTING PROBLEM

In [10], Laporte and Nobert showed that comb inequalities derived for the TSP by Grötschel and Padberg for the TSP [6] can be generalized to the CVRP. In summary, they can be stated as follows.

Let $W_\ell (\ell = 0, \dots, k)$ be subsets of $N - \{1\}$ satisfying

$$(1) \quad |W_\ell - W_0| \geq 1 \quad (\ell = 1, \dots, k)$$

$$(2) \quad |W_\ell \cap W_0| \geq 1 \quad (\ell = 1, \dots, k)$$

$$(3) \quad |W_\ell \cap W_{\ell'}| = 0 \quad (1 \leq \ell < \ell' \leq k)$$

Then, the following condition (comb inequality) holds for every feasible solution of the CVRP:

$$(4) \quad \sum_{\ell=0}^k \sum_{i,j \in W_\ell} x_{ij} \leq |W_0| + \sum_{\ell=1}^k [|W_\ell| - V_1(W_\ell)] - \lfloor \frac{1}{2} k' \rfloor$$

where k' is the number of indices ℓ for which the following condition holds as a strict inequality:

$$V_1(W_\ell - W_0) + V_1(W_\ell \cap W_0) - V_1(W_\ell) \geq 0$$

By property 3 (section 3), k' can also be defined as

$$k' = \sum_{\ell=1}^k [V_1(W_\ell - W_0) + V_1(W_\ell \cap W_0) - V_1(W_\ell)]$$

Using this definition of k' in (4) and substituting V_1 by V_2 , one obtains a tighter form of comb inequality for the CVRP:

$$\sum_{\ell=0}^k \sum_{i,j \in W_\ell} x_{ij} \leq |W_0| + \sum_{\ell=1}^k [|W_\ell| - V_2(W_\ell)] - \left\lfloor \frac{1}{2} \sum_{\ell=1}^k [V_2(W_\ell - W_0) + V_2(W_\ell \cap W_0) - V_2(W_\ell)] \right\rfloor$$

or, equivalently,

$$(5) \quad \sum_{\ell=0}^k \sum_{i,j \in W_\ell} x_{ij} \leq |W_0| + \sum_{\ell=1}^k |W_\ell| - \left\lfloor \frac{1}{2} \sum_{\ell=1}^k [V_2(W_\ell) + V_2(W_\ell - W_0) + V_2(W_\ell \cap W_0)] \right\rfloor$$

This latter expression can also be derived directly from the k -star inequality for the RCVRP expressed in terms of V_2 .

6. FACETIAL PROPERTIES OF COMB INEQUALITIES FOR THE CVRP

It is well known [7] that in the case of the TSP, comb inequalities constitute facets of the polytope of feasible solutions. It is therefore natural to ask whether the same property holds for the CVRP. Unfortunately, the following counter-example shows that this is not the case.

Let $D=2$, $d_i=1$ for every $i \in N - \{1\}$. Let the W_ℓ ($\ell=0, \dots, k$) be subsets of $N - \{1\}$ such that (1), (2), (3) are satisfied. Also assume that k and $|W_0|$ are odd, that $|W_0| > k$ and that $|W_\ell \cap W_0|$, $|W_\ell - W_0|$ are odd for $\ell=1, \dots, k$.

First notice that in this example, $V_1(S) = V_2(S)$ for every non-empty subset S of N . Therefore, (4) and (5) are equivalent. Further, $k' = k$ since for every ℓ , we have

$$\begin{aligned} V_1(W_\ell - W_0) + V_1(W_\ell \cap W_0) - V_1(W_\ell) \\ = \frac{1}{2}(|W_\ell - W_0| + 1) + \frac{1}{2}(|W_\ell \cap W_0| + 1) - \frac{1}{2}|W_\ell| \\ = \frac{1}{2}|W_\ell| - \frac{1}{2}|W_\ell| + 1 > 0 \end{aligned}$$

For every feasible solution, the left-hand side of (4) is bounded above by

$$\left[\frac{1}{2}|W_0| \right] + \frac{1}{2} \sum_{\ell=1}^k |W_\ell| = \frac{1}{2}(|W_0| - 1) + \frac{1}{2} \sum_{\ell=1}^k |W_\ell|$$

while its right-hand side is equal to

$$|W_0| + \frac{1}{2} \sum_{\ell=1}^k |W_\ell| - \left\lceil \frac{1}{2} k \right\rceil = |W_0| + \frac{1}{2} \sum_{\ell=1}^k |W_\ell| - \frac{1}{2}(k+1)$$

Since $|W_0| > k$, the left-hand side of (4) is always strictly smaller than its right-hand side. Hence, the cutting plane defined by (4) (or (5)) does not even intersect the polytope of integer solutions and is therefore not a facet.

However, the question of determining whether some particular forms of comb inequalities (such as connectivity constraints) are facets remains of interest and should constitute the object of further research. Also of interest is the design of efficient search procedures for the detection of violated comb inequalities.

BIBLIOGRAPHY

- [1] DANTZIG, G.B., FULKERSON, D.R., JOHNSON, S.M., "Solution of a Large Scale Travelling Salesman Problem", *Operations Research*, Vol.2, pp.393-410, 1954.
- [2] DANTZIG, G.B., RAMSER, J., "The Truck Dispatching Problem", *Management Science*, Vol.6, pp.81-91, 1959.
- [3] FLEISCHMANN, B., "A Cutting Plane Procedure for the Travelling Salesmen Problem on Road Networks", Research Report QM-02-83, University of Hamburg, 21 pages, 1983.
- [4] FLEISCHMANN, B., "A New Class of Cutting Planes for the Symmetric Travelling Salesman Problem", Research Report QM-03-83, University of Hamburg, 21 pages, 1983.
- [5] GAVISH, B., "The Delivery Problem: New Cutting Planes Procedures", Paper presented at the TIMS XXVI Conference, Copenhagen, June 1984.

- [6] GRÖTSCHEL, M., PADBERG, M.W., "On the Symmetric Travelling Salesman Problem I: Inequalities", Mathematical Programming, Vol.16, pp.265-280, 1979.
- [7] GRÖTSCHEL, M., PADBERG, M.W., "On the Symmetric Travelling Salesman Problem II: Lifting Theorems and Facets", Mathematical Programming, Vol.16, pp.281-302, 1979.
- [8] LAPORTE, G., "An Integer Programming Approach to the Vehicle Scheduling Problem", Cahiers du GERAD G-82-10, Ecole des Hautes Etudes Commerciales de Montréal, 9 pages, 1982.
- [9] LAPORTE, G., NOBERT, Y., "A Branch and Bound Algorithm for the Capacitated Vehicle Routing Problem", Operations Research Spektrum, Vol.5, pp.77-85, 1983.
- [10] LAPORTE, G., NOBERT, Y., "Comb Inequalities for the Vehicle Routing Problem", Methods of Operations Research, Vol.51, pp.271-276, 1984.
- [11] MILIOTIS, P., LAPORTE, G., NOBERT, Y., "Computational Comparison of Two Methods for Finding the Shortest Complete Cycle or Circuit in a Graph", PAIRO (recherche opérationnelle), Vol.15, pp.233-239, 1981.